Markov Models

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A special belief Network   Markov Chain

$S_1 \to S_2 \to S_3 \to S_4 \to \ldots$

We want to reason about time.

Model state of the process by random variable S

dom(S) is possible states of the world.

where is Bob? $S = \{$Lab, classroom, Dorm, taco-Bell$\}$.

markov assumption:- "What happens tomorrow depends only on what happens today and not on what happend in The past"

Markov Models:

A. A. Markov (2nd)

Motivation:

Reason about a process over time

- process can be in one of multiple states

  Bob = $\{$class, library, dorm, lab, cle, taco-bell$\}$

- Markov assumption: what happens next depends only on the current situation, not on what happened before.

Representation: Markov Chain

$B_0 \to B_1 \to B_2 \to B_3 \to B_4$ ← stage.

Assumptions:

- each variable has the same domain
- B_{i+1} is conditionally independent of B_k given B_i : Markov assumption.

$$P(B_{i+1} \mid B_i) = P(B_{i+1} \mid \underbrace{B_0, B_1, B_2, \ldots B_i}_{\text{Independent of past.}})$$

- if $P(B_{i+1} \mid B_i) = P(B_{k+1} \mid B_k)$ for any i and K

stationary Markov chain

We specify a Stationary Markov Chain
- domain (S); state variable S
- $P(S_0)$
- $P(S_{i+1} | S_i)$: dynamics of the system.

Example:

Bob = {class, lab, clc, dorm, taco-bell}

$P(S_0)$

class	0.15
lab	0.25
clc	0.25
dorm	0.20
tb	0.15

$P(S_{i+1} | S_i)$

S_i	class	lab	clc	dorm	tb
class	0.0	0.2	0.0	0.6	0.2
lab	0.1	0.5	0.1	0.1	0.2
clc	0.0	0.0	0.0	0.0	1.0
dorm	0.3	0.1	0.1	0.4	0.1
tb	0.2	0.2	0.2	0.2	0.2

$P(S_4) = ?$ $\quad P(S_4 = tb \mid S_2 = class) = ?$

$P(S_3 \mid S_4 = tb) = ?$

Markov chain is ergodic if for any two states v_1 and v_2 there is a non-zero probability of reaching v_2 from v_1.

Non ergodic $dom(S) = \{a, b, c\}$

$P(S_{i+1} | S_i)$

S_i	a	b	c
a	0.1	0.0	0.0
b	0.0	0.0	0.1
c	0.0	0.1	0.0

state a is not reachable from state b

- Markov chain is periodic if there is a strict temporal regularity in visiting states.

- An ergodic and aperiodic Markov chain has a unique stationary distribution

$$P(S_{i+1}) = P(S_i) \quad \text{as } i \to \infty$$

"equilibrium distribution"

Example:

hyperlink

$$\mathrm{dom}(S) = \{w_0, w_1, w_2, w_3, \ldots, w_k, \ldots, w_n\}$$

$$P(S_0 = w_i) = 1/n \quad \text{for every } 0 \le i \le n$$

Let w_k have n_k links.

$$P(S_{i+1} = w_j \mid S_i = w_k) = \begin{cases} 1/n_k & \text{if } w_k \text{ links to } w_j \\ 1/n & \text{if } w_k \text{ has no links} \\ 0 & \text{otherwize.} \end{cases}$$

Markov chain converges into a Prob. distribution over pages.

(1999 Brim / Page)

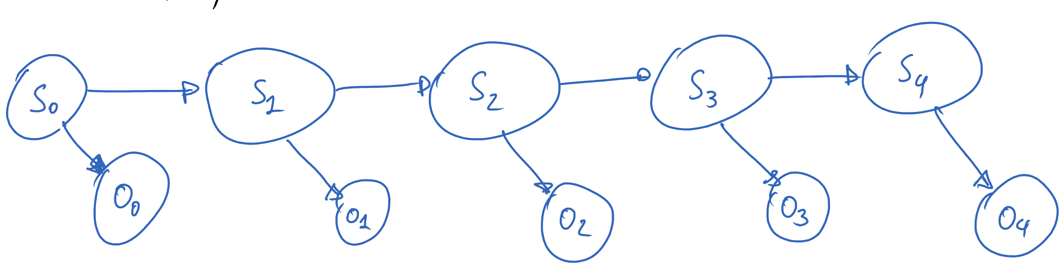
converged after 52 stages; 24 million web pages, 76 million links.

"Page Rank" process. Google Algorithm.

Hidden Markov Model:

Example people report on Bob's location:

$$\mathrm{dom}(O_i) = \{\text{class, dorm, clc, lab, tb}\}$$



Represent Sensors

- $P(S_0)$
- $P(S_{i+1} \mid S_i)$ ← dynamics

- $P(S_{i+1} \mid S_i)$ ← dynamics
- $P(O_i \mid S_i)$ ← sensor model.

$$P(S_1 = tb \mid O_1 = tb) = ?$$

$$P(S_1 = \text{class} \mid O_0 = \text{lab}) = ?$$

- Problem of "filtering": $P(S_i \mid O_0, O_1, O_2, \dots O_i)$
- Problem of "smoothing": $P(S_i \mid O_0, O_1, \dots, O_i, O_{i+1}, \dots O_K)$

How are things computed:

- Markov Chain:

$$S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow$$

Given:
0) $P(S_0)$
1) $P(S_{i+1} \mid S_i)$

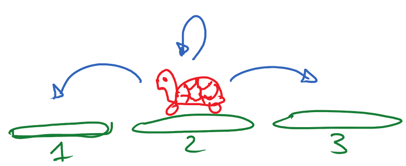
¿ $P(S_k)$?

$$P(S_k) = \sum_{S_{k-1}} P(S_k \wedge S_{k-1})$$

$$= \sum_{S_{k-1}} \underbrace{P(S_k \mid S_{k-1})}_{1)} \underline{P(S_{k-1})}$$

Example:

$S = \{1, 2, 3\}$



$P(S_0) =$

1	0.0
2	1.0
3	0.0

$P(S_{i+1} \mid S_i) =$

S_i \ S_{i+1}	1	2	3
1	0	0.5	0.5
2	0.2	0.4	0.4
3	0.6	0.4	0

where is the turtle at S_2?

$P(S_2)$

$$P(S_1 \wedge S_0) = P(S_2 \mid S_0) P(S_0)$$

S_1	S_0	$P(S_1 \wedge S_0)$
1	1	0 = 0
1	2	1 · 0.2 = 0.2
1	3	0 · 0.5 = 0
2	1	0 = 0
2	2	1 · 0.4 = 0.4
2	3	0 = 0

1	3	$0 \cdot 0.5 = 0$
2	1	$0 = 0$
2	2	$1 \cdot 0.4 = 0.4$
2	3	$0 = 0$
3	1	$0 = 0$
3	2	$2 \cdot 0.4 = 0.4$
3	3	$0 = 0$

$$\sum_{S_0} P(S_1 \wedge S_0) = P(S_1)$$

S_1	$P(S_1)$
1	0.2
2	0.4
3	0.4

$$P(S_2 \wedge S_1) = \underline{P(S_2 \mid S_1)\,P(S_1)}$$

S_i \ S_{i+1}	1	2	3
1	0	0.5	0.5
2	0.2	0.4	0.4
3	0.6	0.4	0

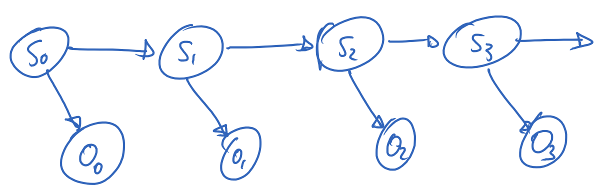
S_2	S_1	$P(S_2 \wedge S_1)$
1	1	$0.2 \times 0 = 0$
1	2	$0.4 \times 0.2 = 0.08$
1	3	$0.4 \times 0.6 = 0.24$
2	1	$0.2 \times 0.5 = 0.10$
2	2	$0.4 \times 0.4 = 0.16$
2	3	$0.4 \times 0.4 = 0.16$
3	1	$0.2 \times 0.5 = 0.10$
3	2	$0.4 \times 0.4 = 0.16$
3	3	$0.4 \times 0 = 0$

$$P(S_2) = \sum_{S_1} P(S_2 \wedge S_1)$$

S_2	$P(S_2)$
1	0.32
2	0.42
3	0.26

- Filtering

Hidden Markov Model



Given
- $P(S_0)$
- $P(S_{i+1} \mid S_i)$
- $P(O_i \mid S_i)$

- $P(S_0 \mid O_0)$?
- $P(S_k \mid O_0, O_1, O_2, \dots O_k)$?

$$P(S_k \mid O_0, O_1 \dots O_k) \xLeftarrow{\text{from}} P(S_{k-1} \mid O_0, O_1, \dots, O_{k-1})$$

$$p(a \mid bc) = \frac{p(a,b,c)}{p(b,c)}$$

$$p(a,b,c) = p(a \mid bc)\,p(b,c)$$

$$P(S_k \mid O_0, O_1 \dots O_k) = P(S_k, O_0, O_1, \dots O_k) \Big/ \sum_{S_k} P(S_k, O_0, O_1, \dots O_k)$$

$$P(O_k \mid S_k, O_0, O_1 \dots O_{k-1})\,P(S_k, O_0, O_1, \dots O_{k-1})$$

$$= \underline{P(O_k \mid S_k)}\,P(S_k, O_0, O_1 \dots O_{k-1})$$

$$= \underbrace{P(O_k|S_k)}\,\underbrace{P(S_k, O_0, O_1 \dots O_{k-1})}$$

$$\to \sum_{S_{k-1}} P(S_k, S_{k-1}, O_0, O_1, \dots O_{k-1})$$

$$= \sum_{S_{k-1}} \underbrace{P(S_k|S_{k-1})}\,\underbrace{P(S_{k-1}, O_0, O_1, \dots O_{k-1})}_{P(S_{k-1}|O_0, O_1 \dots O_{k-1}) \cdot P(O_0, O_1, \dots O_{k-1})}$$

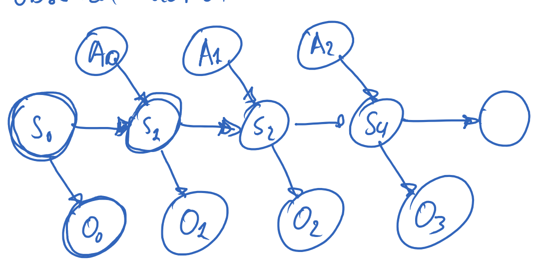
- Example (kind of)

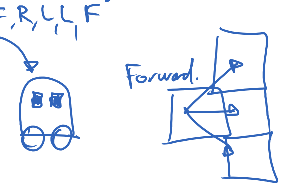
¿ $P(S_3 | O_0, O_1, O_2)$?

Given $P(O_0) \Rightarrow P(S_0|O_0)$, $P(O_1)$ $\Rightarrow P(S_1|O_0, O_1)$, $P(O_2)$ $\Rightarrow P(S_3|O_0, O_1, O_2)$

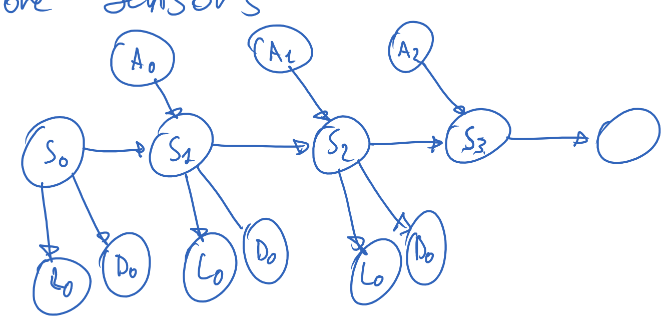
Hidden Markov Model w/ Actions.

Add an observed action to the chain





- More Sensors



$P(L_i|S_i)$
$P(D_i|S_i)$

$$\underbrace{P(S_k | L_0 D_0 L_1 D_1 \dots L_k D_k)}_{\text{Sensor Fusion}}$$

Next Class :- Include Actions and Preferences.

$\Rightarrow$ "Markov Decision Processes"

$\Rightarrow$ "Reinforcement Learning"